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Algorithms and Probability **Session Sheet S8** **FS 2026**

For the exercise sessions on 16 April 2026.

Exercise S8.1 – *Inequalities*

We throw 1000 fair coins and denote the results by $C_1, \dots, C_{1000}$. We want to count the number of neighbouring coins that both show "heads". Two neighbouring coins are coins of the form C_i, C_{i+1} or C_{1000}, C_1 (imagine the coins being placed on a circle). Let X denote the number of neighbouring coins that both show "heads". We will use different inequalities to bound the probability that X is significantly larger than its expected value. Make sure to check all relevant conditions before applying an inequality.

(a) Show that $\mathbb{E}[X] = 250$.

(b) Use Markov's inequality to bound $\Pr[X \geq 300]$.

(c) Compute $\mathrm{Var}[X]$ and use Chebychev's inequality to bound $\Pr[X \geq 300]$.

(d) We define Y as the number of neighbouring coins that both show "heads" and for which the first coin has an odd index. That is, we only consider pairs of the form X_{2i-1}, X_{2i}, with $i = 1, \dots, 500$. Show that $\mathbb{E}[Y] = 125$ and use Chernoff's bound for $\Pr[Y \geq 150]$.

(e) Use (d) to bound $\Pr[X \geq 300]$.
Remark: Could we just apply Chernoff's bound right away?

Solution S8.1 – *Inequalities*

Let $n = 1000$. Whenever an index would be $n+1$ resp. $n+2$, consider the index 1 resp. 2 instead.

a) For $i \in [n]$ let X_i be the indcator variable of the event that both C_i and C_{i+1} show "heads". By linearity of expectation, we get

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \Pr[X_i = 1] = \frac{n}{4} = 250 \ .$$

b) Because $X \geq 0$, Markov's inequality implies

$$\Pr(X \geq 300) \leq \frac{\mathbb{E}[X]}{300} = \frac{5}{6} \ .$$

c) To compute $\mathrm{Var}[X]$, we first compute the expected value of $X^2 = \left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{(i,j)\in[n]^2} X_i X_j$. To do so, we split the latter sum into three sums: Let $S_0, S_1, S_{\geq 2}$ be the sets of indices

$(i,j) \in [n]^2$ with $|i-j| = 0$, $|i-j| = 1$, $|i-j| \geq 2$. For $(i,j) \in S_0$, we have $X_i X_j = 1$ if and only if both C_i and C_{i+1} show "heads" (and zero otherwise). This happens with probability $\frac{1}{4}$.
For $(i,j) \in S_1$, we have $X_i X_j = 1$ if and only if three specific coins all show heads. This happens with probability $\frac{1}{8}$.
For $(i,j) \in S_{\geq 2}$, we have $X_i X_j = 1$ if and only if four specific coins all show heads. This happens with probability $\frac{1}{16}$.
The set S_0 contains n elements, the set S_1 contains $2n$ elements (because we consider ordered pairs!), and the set $S_{\geq 2}$ contains the remaining $n^2 - n - 2n = n^2 - 3n$ elements.

Now we are ready to compute $\mathbb{E}[X^2]$:

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{(i,j)\in[n]^2} X_i X_j \\
&= \sum_{(i,j)\in S_0} X_i X_j + \sum_{(i,j)\in S_1} X_i X_j + \sum_{(i,j)\in S_{\geq 2}} X_i X_j \\
&= \frac{1}{4} \cdot n + \frac{1}{8} \cdot 2n + \frac{1}{16} \cdot (n^2 - 3n) \\
&= \frac{n^2}{16} + \frac{5n}{16}\,.
\end{aligned}$$

Thus,

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n^2}{16} + \frac{5n}{16} - \frac{n^2}{16} = \frac{5n}{16}\,.$$

Knowing the variance of X, we can use Chebychev's inequality to bound $\Pr[X \geq 300]$ as follows.

$$\begin{aligned}
\Pr[X \geq 300] &\leq \Pr[\,|X - \mathbb{E}[X]| \geq 50] \\
&\leq \frac{\mathrm{Var}[X]}{50^2} \\
&= \frac{5000/16}{2500} = \frac{1}{8}
\end{aligned}$$

d) By definition, we have $Y = \sum_{i=1}^{500} X_{2i-1}$. By linearity of expectation, we have $\mathbb{E}[Y] = 125$. Because the variables X_i with odd indices are independent, Y is the sum of independent Bernouuli random variables. Hence, the Chernoff bound applies. Using $\delta = \frac{1}{5}$, we get

$$\Pr[Y \geq 150] = \Pr[Y \geq (1+\delta)\mathbb{E}[Y]) \leq e^{-\frac{\delta^2 \mathbb{E}[Y]}{3}} = e^{-5/3}\,.$$

e) We cannot directly apply the Chrenoff bound to X because the X_i are not independent. However for $Z = \sum_{i=1}^{500} X_{2i}$, we can use the same arguments as in (d) to obtain $\Pr[Z \geq 150] \leq e^{-\frac{5}{3}}$. For $X = Y + Z$ to be larger than 300, one of Y and Z has to larger than 150. Hence, combining (d) and the union bound, we get

$$\Pr[X \geq 300] \leq \Pr[Y \geq 150 \;\vee\; Z \geq 15] \leq \Pr[Y \geq 150] + \Pr[Z \geq 150] \leq 2e^{-5/3} < 0.38$$