

For the exercise sessions on 26 March 2026.

**Exercise S6.1 – Probability Space**

- (a) For each of the following subtasks, either define a probability space and events  $A$  and  $B$  (and  $C$ ) with the described properties, or prove that such a space cannot exist. Make sure that you define both, the sample space (“Ergebnismenge”)  $\Omega$  and the probabilities of the atomic events (“Elementarereignisse”).
- (i)  $\Pr[A] = \frac{1}{4}$ ,  $\Pr[B] = \frac{1}{3}$  and  $\Pr[A \cup B] = \Pr[A] + \Pr[B]$ .
  - (ii)  $\Pr[A] = \frac{1}{4}$ ,  $\Pr[B] = \frac{1}{3}$  and  $\Pr[A \cup B] < \Pr[A] + \Pr[B]$ .
  - (iii)  $\Pr[A] = \Pr[B]$ ,  $\Pr[A \cap B] = \frac{1}{4}$ , and  $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$  (that is  $A$  and  $B$  are independent).
  - (iv)  $\Pr[A] = \Pr[B] = \Pr[C] = \frac{5}{6}$  and  $\Pr[A \cap B \cap C] = 0$ .
- (b) Samantha has a fair, six-sided die and a 5 CHF coin. She rolls the die and tosses the coin. Samantha considers her experiment a success if the coin shows a strictly larger value than the die (for the coin, heads is counted as 0; tails is counted as 5). Model her experiment with a suitable probability space. Explicitly define the event  $A$  that the experiment is a success and determine its probability  $\Pr[A]$ .
- (c) Oliver owns three pairs of shoes – two blue pairs, and one yellow, which he stores unordered in his wardrobe. One morning, during a power outage, he has to put on his shoes in complete darkness. He randomly (uniformly at random) grabs two shoes from the wardrobe and tries to put them on.

We let  $A$  denote the event that he picked one left shoe and one right shoe (i.e. he is able to put on the shoes he picked), and we let  $B$  be the event that the two shoes he picked have the same color.

Model this setting as a probability space and compute  $\Pr[A]$  and  $\Pr[A|B]$ .

**Solution S6.1 – Probability Space**

- (a) There are multiple ways of defining suitable probability spaces. For (i), (ii), and (iii) we can choose the probability space of a fair 12 sided die. That is,  $\Omega = \{1, 2, \dots, 11, 12\}$  and  $\Pr[\omega] = \frac{1}{12}$  for all  $\omega \in \Omega$ . It remains to define the events  $A$  and  $B$ .
- (i) Let  $A = \{1, 2, 3\}$  and  $B = \{4, 5, 6, 7\}$ . Then  $\Pr[A] = \frac{3}{12} = \frac{1}{4}$ ,  $\Pr[B] = \frac{4}{12} = \frac{1}{3}$  and  $\Pr[A \cup B] = \Pr[\{1, 2, \dots, 7\}] = \Pr[A] + \Pr[B]$ .
  - (ii) Let  $A = \{4, 8, 12\}$  and  $B = \{3, 6, 9, 12\}$ . Then  $\Pr[A] = \frac{1}{4}$ ,  $\Pr[B] = \frac{1}{3}$  (as in (i)) and  $\Pr[A \cup B] = \Pr[\{3, 4, 6, 8, 9, 12\}] = \frac{6}{12} < \frac{7}{12} = \Pr[A] + \Pr[B]$ .

- (iii) Let  $A := \{1, 2, 3, 4, 5, 6\}$  and  $B := \{2, 4, 6, 8, 10, 12\}$ . Then  $\Pr[A] = \frac{6}{12} = \frac{1}{2}$  and  $\Pr[B] = \frac{6}{12} = \frac{1}{2}$  and  $\Pr[A \cap B] = \Pr[\{2, 4, 6\}] = \frac{3}{12} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr[A] \cdot \Pr[B]$ .
- (iv) There is no probability space for which such events exist. To prove this, we use the Union bound and the fact that  $\Pr[\bar{A}] = 1 - \Pr[A]$ . Assume that was a probability space and events  $A$ ,  $B$ , and  $C$  that satisfy  $\Pr[A] = \Pr[B] = \Pr[C] = \frac{5}{6}$  and  $\Pr[A \cap B \cap C] = 0$ . Then

$$\begin{aligned} 1 &= 1 - \Pr[A \cap B \cap C] \\ &= \Pr[\bar{A} \cup \bar{B} \cup \bar{C}] \\ &\stackrel{U.B.}{\leq} \Pr[\bar{A}] + \Pr[\bar{B}] + \Pr[\bar{C}] = \frac{3}{6} < 1 \end{aligned}$$

is a contradiction.

- (b) We can model the experiment as follows. As our sample space we choose  $\Omega := \{1, 2, \dots, 6\} \times \{0, 5\}$ . For every  $(a, b) \in \Omega$ , the first entry  $a$  stands for the result of the die, while the second entry  $b$  represents the result of the coin. Because the coin and the die are independent Laplacian experiments, also our space is Laplacian, namely  $\Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{12}$  for all  $\omega \in \Omega$ .

The event that Samantha considers the experiment a success can be described as follows:  $A := \{(1, 5), (2, 5), (3, 5), (4, 5)\}$ . Thus,  $\Pr[A] = \frac{4}{12} = \frac{1}{3}$ .

- (c) Let  $S = \{b_l, b_r, b_l^*, b_r^*, y_l, y_r\}$  be Oliver's shoes. The indices mark if a shoe is a left shoe or a right shoe.

Let  $\Omega := \binom{S}{2}$  be the set of all unordered pairs of the shoes. Because each pair of shoes is equally likely to be picked, we have a uniform distribution. That is, for all  $\omega \in \Omega$  we have  $\Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{15}$ .

The events  $A$  and  $B$  can be described as subsets of  $\Omega$ :

$$\begin{aligned} A &:= \{\{y_l, y_r\}, \{y_l, b_r\}, \{y_l, b_r^*\}, \{b_l, y_r\}, \{b_l, b_r\}, \{b_l, b_r^*\}, \{b_l^*, y_r\}, \{b_l^*, b_r\}, \{b_l^*, b_r^*\}\}, \\ B &:= \{\{y_l, y_r\}, \{b_l, b_r\}, \{b_l, b_l^*\}, \{b_l, b_r^*\}, \{b_r, b_l^*\}, \{b_r, b_r^*\}, \{b_l^*, b_r^*\}\}, \\ A \cap B &:= \{\{y_l, y_r\}, \{b_l, b_r\}, \{b_l, b_r^*\}, \{b_l^*, b_r\}, \{b_l^*, b_r^*\}\} \end{aligned}$$

Hence,  $\Pr[A] = \frac{9}{15} = \frac{3}{5}$  and  $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{5/15}{7/15} = \frac{5}{7}$ .