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Algorithms and Probability **Session Sheet S10** **FS 2026**

For the exercise sessions on 30 April 2026.

Exercise S10.1 – *Probabilistic method*

So far, we mainly used randomization to construct efficient algorithms. However, randomization can also be used to prove statements that don't involve randomness themselves. In fact, such proves can usually be phrased by describing a Monte-Carlo algorithm. Answer the two following questions. If you want to, you can describe your solution as a Monte-Carlo algorithm.

(a) You are given 1000 subsets $A_1, \ldots, A_{1000}$ of $\{1, \ldots, n\}$. Each subset has size at least 11. Can the numbers $1, \ldots, n$ be colored 'red' and 'blue', such that each set A_i contain at least one 'red' and at least one 'blue' number?

(b) You are given 10 points in the plane (i.e., $\mathbb{R}^2$). Can you place (filled) disks of radius 1 in the plane, such that (i) each of the 10 points is contained in a disk, and (ii) all disks are disjoint (i.e., their centers have distance at least 2)?
Remark: You may cover multiple points with the same disk. In particular, the task would be trivial if all points are very close to each other (then you only need one disk to cover all of them). Similarly, if all points are very far apart, you could use one disk per point without having to worry about disjointness.
Remark: a formal solution to this exercise would be very technical. Focus more on the ideas than on technical details

Solution S10.1 – *Probabilistic method*

(a) For each of the numbers $1, \ldots, n$ sample one of the colors 'red' and 'blue' uniformly at random. We claim that there is a positive probability, that this results in a feasible coloring. Note that a coloring is feasible, if no set is entirely colored with one color. First, fix an index i. The probability that the set A_i is unicolor can be bounded as follows.

$$\begin{aligned}\Pr[A_i \text{ is unicolor}] &= \Pr[A_i \text{ is completely red}] + \Pr[A_i \text{ is completely blue}] \\ &= \frac{1}{2^{|A_i|}} + \frac{1}{2^{|A_i|}} \\ &\leq 2 \cdot \frac{1}{2^{11}} = \frac{1}{2^{10}}\,.\end{aligned}$$

Hence, the probability, that at least one set is unicolor, can be upper bounded as follows (using the union bound):

$$\begin{aligned}\Pr[\text{at least one unicolor set}] &\leq \sum_{i=1}^{1000} \Pr[A_i \text{ is unicolor}] \\ &\leq 1000 \cdot \frac{1}{2^{10}} < 1.\end{aligned}$$

This implies Pr[feasible coloring] $= 1 -$ Pr[at least one unicolor set] > 0. Thus, by sampling the colors, we have a positive probability to find a feasible coloring. This in particular implies that there always exists a feasible coloring!

(b) We make our task even harder: Assume that the ten points are fixed, but they are not told to you. The best thing you could do in this setting would be trying to cover as much space as possible. So first, let us think about what fraction of $\mathbb{R}^2$ we can cover with disjoint unit disks (this is already a bit imprecise because this fraction would be $\frac{\cdots}{\infty}$). There is a straight forward way to pack disks very dense into the plane. In particular, we can cover a

$$\frac{\pi/2}{\frac{1}{2} \cdot 2 \cdot \sqrt{2^2 - 1^2}} = \frac{\pi\sqrt{3}}{6} > 0.906$$

fraction of the plane. (This can e.g. be seen, by covering the plane with equilateral triangles of side length 2. When putting a unit disk around each corner of the triangles, then each in each triangle, the covered area is $\pi/2$, while the size of the triangle is $\frac{1}{2} \cdot 2 \cdot \sqrt{2^2 - 1^2}$.)

So far, we ensured that we can cover more than 90% of the plane with unit disks. However, if we deterministically choose the disks, we could hit any number of the 10 points. However, if we randomly shift our disk-pattern up/down and left/right, each point in the plane has the same probability of $\frac{\pi\sqrt{3}}{6}$ of being covered (this is again not very formal: technically we would have to specify how we choose this random shift). For a randomly shifted such disk-pattern, let X_i (for $i = 1, \ldots, 10$) be the random variable indicating if the i-th point is covered. Then for every i we have $\mathbb{E}[X_i] = \Pr[X_i = 1] > 0.9$. Thus, by linearity of expectation

$$\mathbb{E}\left[\sum_{i=1}^{10} X_i\right] > 9\,.$$

Hence, there is a configuration for which more than 9 (i.e., all ten) of the points are covered.

To summarize: even if we don't know where the points are, we have a positive probability to cover them all. In particular, there always is a ways to cover them all.