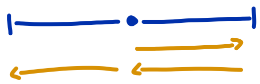
# Quiz

Consider the pasture break problem, where the distance $k$ to the gap in the fence is known in advance. There is an algorithm that always finds this gap in at most $3k$ steps.

- [x] True
- [ ] False



Consider the pasture break problem, where the distance $k$ to the gap in the fence is ***not*** known in advance. There is an algorithm that always finds this gap in at most $3k$ steps.

- [ ] True
- [x] False (from lecture)

Consider the pasture break problem, where the distance $k$ to the gap in the fence is ***not*** known in advance. Then, it is an efficient strategy to

- [ ] ...walk as far to the left as possible until the gap is found.
- [ ] ...walk as far to the right as possible until the gap is found.
- [x] ...walk in a 'zick-zack' until the gap is found.

} we will never find it if we choose the wrong direction

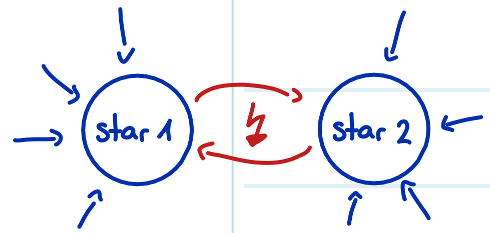
Consider the problem of finding a star $p_s$ among $n \geq 2$ people $p_1, \ldots, p_n$. There is an algorithm that finds a star if one exists, using at most $O(n)$ questions (meaning that there is a constant $C$ such that the algorithm uses at most $C \cdot n$ questions).

- [x] True from lecture: possible with at most $3n-4 \leq O(n)$ questions
- [ ] False

Consider the problem of finding a star $p_s$ among $n \geq 2$ people $p_1, \ldots, p_n$. Exactly one of the following statements is always true. Choose the correct one.

- [ ] There always exists exactly one star among these people.
- [x] There always exists either exactly one or no star among these people.
- [ ] There may exist two distinct stars among these people.
- [ ] There always exists no star among these people.

= never exists a star




arrow = "knows"

Recall: a star knows nobody else and is known by everybody else

$\Rightarrow$ can't have two stars

Consider the problem of finding a star $p_s$ among $n \geq 2$ people $p_1, \ldots, p_n$. Assume that there exists a star. With your first question you find out that $p_1$ knows $p_2$. Which of the following strategies will always correctly find the star?

$\to$ we only need the "elimination phase", i.e if only one candidate is left, he must be a star

$p_1 \xrightarrow{\text{knows}} p_2$ thus $p_1$ can't be a star

- [x] Completely disregard the person $p_1$, and continue to recursively find a star among $p_2, \ldots, p_n$.
- [ ] Completely disregard the person $p_2$, and continue to recursively find a star among $p_1, p_3, p_4, \ldots, p_n$.
- [ ] Neither of the other options, to find a star we also need to know whether or not $p_2$ knows $p_1$.

Let $A(n)$ be a mathematical statement (for $n \in \mathbb{N}$). Assume that we know the following:

- $A(2)$ is true, and
- for every $n \in \mathbb{N}$, if $A(n)$ is true, then also $A(n+1)$ is true.

Which of the following are then necessarily true? Select all options that are necessarily true.

- [ ] $A(1)$ is true.
- [ ] $A(n)$ is true for all $n \in \mathbb{N}$. no, not necessarily for $n = 1 \in \mathbb{N}$
- [x] $A(3)$ is true.

1 2 3 4 5 ...

For all real numbers $x \geq -1$ and positive integers $n \in \mathbb{N}$, the following holds: $(1+x)^n \geq 1+nx$.

- [x] True
- [ ] False

Bernoulli's inequality from lecture

$n^2 + 2n \leq O(n^2)$.

- [x] True
- [ ] False

$$\lim_{n\to\infty} \frac{n^2+2n}{n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} + \frac{2\cancel{n}}{n^{\cancel{2}}} = \lim_{n\to\infty} 1 + \underbrace{\frac{2}{n}}_{\to 0} = 1$$

anderer Lösungsweg: $n^2+2n \underbrace{\leq}_{\text{weil } n \geq 1} n^2+2n^2 = 3n^2 \leq O(n^2)$

$1+2+3+\cdots+n \leq O(n)$.

- [ ] True
- [x] False

$1+2+\ldots+n = \frac{n(n+1)}{2}$ by induction

$$\lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n} = \lim_{n\to\infty} \frac{\cancel{n}(n+1)}{2\cancel{n}} = \lim_{n\to\infty} \frac{n+1}{2} = \infty \implies \text{false}$$

Enter the smallest positive integer $k \in \mathbb{N}$ such that $2n + \underbrace{10n^5}_{\text{only largest term matters}} + 2 \cdot (\log n)^3 \leq O(n^k)$.

Answer: 5

$$\lim_{n\to\infty} \frac{2n+10n^5+2(\log n)^3}{n^5} = \lim_{n\to\infty} \underbrace{\frac{2\cancel{n}}{n^{\cancel{5}4}}}_{\to 0} + \frac{10\cancel{n^5}}{\cancel{n^5}} + \underbrace{\frac{2(\log n)^3}{n^5}}_{\to 0} = 10$$

because $\frac{2(\log n)^3}{n^5} \leq \frac{2n^3}{n^5} = \frac{2}{n^2} \to 0$

The function $f(n) = 100^n$ ($f : \mathbb{N} \to \mathbb{R}^+$) grows asymptotically faster than the function $g(n) = 99^n$ ($g : \mathbb{N} \to \mathbb{R}^+$). Recall that we say that $f$ grows asymptotically faster than $g$ if $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

- [x] True
- [ ] False

$$\lim_{n\to\infty} \frac{99^n}{100^n} = \lim_{n\to\infty} \left(\frac{99}{100}\right)^n = 0$$

weil $\frac{99}{100} = 0.99 < 1$

# Discussion Exercise Sheet 1

**Exercise 1.1** *Sum of Squares* (**2 bonus points**). → Peergrading Aufgabe für heute

Prove by mathematical induction that for every positive integer $n$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Recall the notation used here: $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2$.

wir iterieren über die Variable k

Base Case: Sei $k = 1$

$\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2\,1+1)}{6}$ stimmt, deshalb gilt die Aussage für $k = 1$

Induktions-Hypothese: Wir nehmen an, dass die Aussage für ein beliebiges $k \in \mathbb{N}$ gilt,

d h $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

Induktions-Schritt: Wir wollen die Aussage für $k+1$ zeigen, d h $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \underbrace{\sum_{i=1}^{k} i^2} + (k+1)^2 \\
&\overset{IH}{=} \underbrace{\frac{k(k+1)(2k+1)}{6}} + (k+1)^2 && \text{mit 6 erweitern} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \text{auf einen Bruch nehmen} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && (k+1) \text{ ausklammern} \\
&= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6} && \text{ausmultiplizieren} \\
&= \frac{(k+1)\left(2k^2 + k + 6k + 6\right)}{6} \\
&= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} && \text{faktorisieren} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{mit } (k+1) \text{ formulieren} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \blacksquare
\end{aligned}$$

**Exercise 1.2** *Sum of reciprocals of roots*

Consider the following claim:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \le \sqrt{n}.$$

A student provides the following induction proof. Is it correct? If not, explain where the mistake is.

**Base case:** $n = 1$,

$$\frac{1}{\sqrt{1}} \le 1, \text{ which is true.}$$

correct

**Induction hypothesis:** Assume the claim holds for $n = k$, i.e.

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \sqrt{k}.$$

correct

**Induction step:** Then, starting from the claim we need to prove for $n = k + 1$ and using logical equivalences:

but it doesn't start from the claim

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \le \sqrt{k+1} \overset{\checkmark\ \checkmark}{\iff} \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \sqrt{k+1} - \frac{1}{\sqrt{k+1}}$$

$$\overset{\checkmark\ \checkmark}{\iff} \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \frac{k+1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+1}}$$

korrekte Umformungen

$$\overset{\checkmark}{\cancel{\iff}} \underbrace{\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \frac{k}{\sqrt{k+1}}} \le \frac{k}{\sqrt{k}} \le \sqrt{k},$$

I.H.

which is true, therefore the claim holds by the principle of mathematical induction.

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \frac{k}{\sqrt{k+1}} \cancel{\Longleftarrow} \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \frac{k}{\sqrt{k}}$$

$$\Longleftarrow \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le \sqrt{k}$$

I.H.

Induktionsschritt geht in die falsche Richtung: $k+1 \to k$

**Exercise 1.3** *Asymptotic growth – part i* **(2 bonus points)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N}_{\geq 10} \to \mathbb{R}^+$ are two functions, then:

- We say that $f$ *grows asymptotically slower than* $g$ if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that $g$ *grows asymptotically faster than* $f$.

Prove or disprove each of the following statements.

(a) $f(m) = 100m^3 + 10m^2 + m$ grows asymptotically slower than $g(m) = 0.001 \cdot m^5$.

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{100m^3 + 10m^2 + m}{0.001m^5} = \lim_{m\to\infty} \underbrace{\frac{100\cancel{m^3}}{0.001m^{\cancel{5}2}}}_{\to 0} + \underbrace{\frac{10\cancel{m^2}}{0.001m^{\cancel{5}3}}}_{\to 0} + \underbrace{\frac{\cancel{m}}{0.001m^{\cancel{5}4}}}_{\to 0} = 0 \quad \Rightarrow \text{True}$$

(b) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = (\log m)^3$.

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{\log(m^3)}{(\log m)^3} = \lim_{m\to\infty} \frac{3\cancel{\log m}}{(\log m)^{\cancel{3}2}} = \lim_{m\to\infty} \frac{3}{(\log m)^2} = 0 \quad \Rightarrow \text{True}$$

log Gesetz

(c) $f(m) = e^{2m}$ grows asymptotically slower than $g(m) = 2^{3m}$.

***Hint:*** *Recall that for all $n, m \in \mathbb{N}$, we have $n^m = e^{m \ln n}$.* (very bad hint)

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{e^{2m}}{2^{3m}} = \lim_{m\to\infty} \frac{(e^2)^m}{(2^3)^m} = \lim_{m\to\infty} \left(\frac{e^2}{2^3}\right)^m = 0 \quad \Rightarrow \text{True}$$

Potenzgesetz Potenzgesetz weil $\frac{e^2}{2^3} \approx 0.92 < 1$

**Exercise 1.4** *Asymptotic growth – part ii.*

Prove or disprove each of the following statements.

(a)* If $f(m), g(m) > 1$ and $f(m)$ grows asymptotically slower than $g(m)$, then $\log(f(m))$ grows asymptotically slower than $\log(g(m))$.

False Counterexample: let $f(m) = m$, $g(m) = m^2$

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{m}{m^2} = \lim_{m\to\infty} \frac{1}{m} = 0 \quad \Rightarrow f \text{ grows slower than } g$$

$$\lim_{m\to\infty} \frac{\log(f(m))}{\log(g(m))} = \lim_{m\to\infty} \frac{\log(m)}{\log(m^2)} = \lim_{m\to\infty} \frac{\log(m)}{2\log(m)} = \frac{1}{2} \quad \Rightarrow \log(f) \text{ and } \log(g) \text{ have same growth}$$

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}\leq\frac{1}{\sqrt{3n+1}},\quad n\geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

**Base Case:** Sei $k=1$ $\quad \frac{1}{2}=\frac{1}{\sqrt{4}}=\frac{1}{\sqrt{3\cdot 1+1}}$ stimmt, deshalb gilt die Aussage für $k=1$

**Induktions-Hypothese:** Wir nehmen an, dass die Aussage für ein beliebiges $k\in\mathbb{N}$ gilt,

d.h. $\frac{1}{2}\ \frac{3}{5}\ ..\ \frac{2k-1}{2k}\leq\frac{1}{\sqrt{3k+1}}$

**Induktions-Schritt:** Wir wollen die Aussage für $k+1$ zeigen, d.h. $\frac{1}{2}\ \frac{3}{5}\ ..\ \frac{2(k+1)-1}{2(k+1)}\leq\frac{1}{\sqrt{3(k+1)+1}}$

$$\underline{\frac{1}{2}\ \frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k}}\cdot\frac{2k+1}{2k+2}\overset{\text{I.H.}}{\leq}\underline{\frac{1}{\sqrt{3k+1}}}\cdot\frac{2k+1}{2k+2}$$

$$\vdots$$

$$\leq\frac{1}{\sqrt{3(k+1)+1}}$$

wollen wir zeigen durch logische Äquivalenzen

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2}\leq\frac{1}{\sqrt{3(k+1)+1}}$$

mal $\sqrt{3k+1}$

$$\Leftrightarrow\quad \frac{2k+1}{2k+2}\leq\frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$

Potenzgesetz

$$\Leftrightarrow\quad \frac{2k+1}{2k+2}\leq\sqrt{\frac{3k+1}{3k+4}}$$

beide Seiten quadrieren

$$\Leftrightarrow\quad \frac{(2k+1)^2}{(2k+2)^2}\leq\frac{3k+1}{3k+4}$$

$$\Leftrightarrow\quad (2k+1)^2(3k+4)\leq(2k+2)^2(3k+1)$$

$$\Leftrightarrow\quad (4k^2+4k+1)(3k+4)\leq(4k^2+8k+4)(3k+1)$$

$$\Leftrightarrow\quad 12k^3+12k^2+3k+16k^2+16k+4\leq 12k^3+24k^2+12k+4k^2+8k+4$$

$$\Leftrightarrow\quad \cancel{12k^3}+\cancel{28k^2}+19k+\cancel{4}\leq\cancel{12k^3}+\cancel{28k^2}+20k+\cancel{4}$$

minus $19k$

$\Leftrightarrow\quad 0\leq k$ $\qquad$ Das bedeutet, der I.S. ist gültig für alle $k\geq 0$ $\blacksquare$

# O-Notation

f und g sind Funktionen von N nach $\mathbb{R}^+$

**Definition 1** ($O$-Notation)**.** For $f : N \to \mathbb{R}^+$, es existiert eine positive Konstante C

$$O(f) := \{g : N \to \mathbb{R}^+ \mid \exists C > 0\, \forall n \in N\; g(n) \le C \cdot f(n)\}.$$

Definitionszeichen

bedingt dass

sodass für alle Eingaben $n \in N$ gilt dass

g ist höchstens um einen konstanten Faktor grösser als f

- O(f) ist die Menge aller Funktionen g, die asymptotisch höchstens so schnell wie f wachsen
- Was f und g am Anfang machen (für kleine n) machen ist egal
- Wir ignorieren konstante Faktoren
- Beispiel: $f(n) = n$, $g(n) = 3n$, dann gilt $g \in O(f)$ mit $C = 3$
- Wir schreiben oft "$\le$" statt "$\in$"

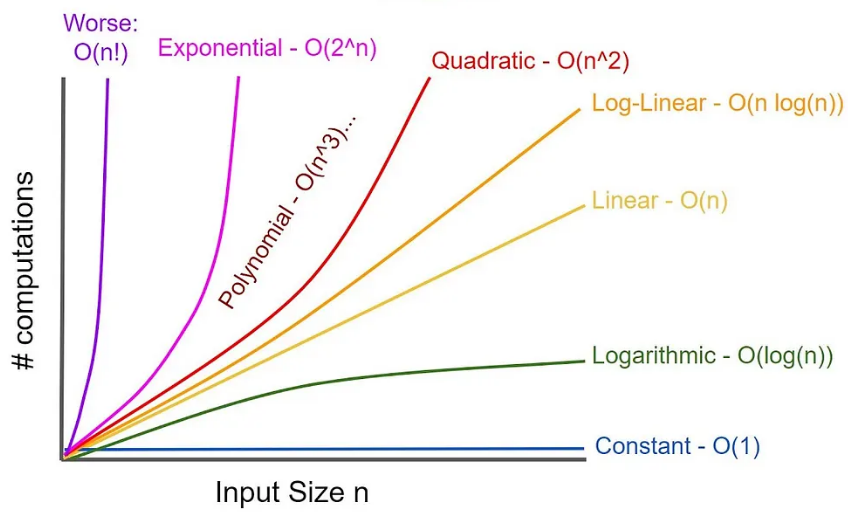
**Theorem 1.** *Let $N$ be an infinite subset of $\mathbb{N}$ and $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \le O(g)$ and $g \not\le O(f)$.* g wächst schneller
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, then $f \le O(g)$ and $g \le O(f)$.* beide wachsen gleich schnell
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \not\le O(g)$ and $g \le O(f)$.* f wächst schneller

The following theorem can also be helpful when working with $O$-notation.

**Theorem 2.** *Let $f, g, h : N \to \mathbb{R}^+$. If $f \le O(h)$ and $g \le O(h)$, then*

1. *For every constant $c > 0$, $c \cdot f \le O(h)$.* "konstante Faktoren ignorieren"
2. *$f + g \le O(h)$.* Bsp. $\log n \le O(n)$ und $2n \le O(n)$ $\Rightarrow$ $\log n + 2n \le O(n)$




## Aufgaben:

Prove or disprove: $3n \leq O(n \log n)$

$$\lim_{n\to\infty} \frac{3\cancel{n}}{\cancel{n}\log n} = \lim_{n\to\infty} \frac{3}{\log n} = 0$$

Per Theorem 1 folgt: $3n \leq O(n \log n)$ ■

Prove or disprove: $\log_3(n^3) \leq O(\ln n)$

$$\lim_{n\to\infty} \frac{\log_3(n^3)}{\ln n} \underset{\text{log Gesetz}}{=} \lim_{n\to\infty} \frac{3\log_3(n)}{\ln n} \underset{\text{log Gesetz}}{=} \lim_{n\to\infty} \frac{3\,\frac{\cancel{\ln(n)}}{\ln(3)}}{\cancel{\ln(n)}} = \lim_{n\to\infty} \frac{3}{\ln(3)} = \frac{3}{\ln(3)}$$

Per Theorem 1 folgt: $\log_3(n^3) \leq O(\ln n)$ ■

Prove or disprove: $2^n \leq O(n^3)$

$$\lim_{n\to\infty} \frac{2^n}{n^3} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{\ln(2)\ 2^n}{3n^2} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{\ln(2)\ \ln(2)\ 2^n}{6n} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{\ln(2)\ \ln(2)\ \ln(2)\ 2^n}{6} = \infty$$

Per Theorem 1 folgt: $2^n \not\leq O(n^3)$, was die Aussage widerlegt

Prove or disprove: $\log_2(n) \cdot \sqrt{n} \leq O(\ln n)$

$$\lim_{n\to\infty} \frac{\log_2(n)\,\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\frac{\cancel{\ln n}}{\ln 2}\,\sqrt{n}}{\cancel{\ln n}} = \lim_{n\to\infty} \frac{1}{\ln 2}\cdot\sqrt{n} = \infty$$

log Gesetz

Per Theorem 1 folgt: $\log_2(n)\,\sqrt{n} \notin O(\ln n)$, was die Aussage widerlegt

Prove or disprove: $2^{n \log_2 n} \leq O(2^{n \ln n})$

$$\lim_{n\to\infty} \frac{2^{n\log_2(n)}}{2^{n\ln(n)}} = \lim_{n\to\infty} 2^{\underbrace{n\log_2(n) - n\ln(n)}_{\to\infty}} = \infty$$

müssen wir noch zeigen

$$\lim_{n\to\infty} n\log_2(n) - n\ln(n) = \lim_{n\to\infty} n\frac{\ln(n)}{\ln(2)} - n\ln(n) = \lim_{n\to\infty} \underbrace{n\ln(n)}_{\to\infty}\underbrace{\left(\frac{1}{\ln(2)} - 1\right)}_{\approx 0.44} = \infty$$

log Gesetz

Per Theorem 1 folgt: $2^{n\log_2(n)} \notin O(2^{n\ln(n)})$, was die Aussage widerlegt

Order the following functions by their asymptotic growth rate as $n \to \infty$:

$$n!,\quad \sqrt{n},\quad 2^n,\quad 1,\quad n^n,\quad \log n,\quad n,\quad n^{\frac{n}{2}},\quad n^2$$

$O(1) \leq O(\log n) \leq O(\sqrt{n}) \leq O(n) \leq O(n^2) \leq O(2^n) \leq O(n^{\frac{n}{2}}) \leq O(n!) \leq O(n^n)$

Order the following functions:

$$n^{10},\quad n\log n,\quad 3^n,\quad \sqrt{n}$$

$O(\sqrt{n}) \leq O(n\log n) \leq O(n^{10}) \leq O(3^n)$

Order the following functions:

$$n^2+\log n, \quad n+100, \quad \sqrt{n}+\log^2 n, \quad n^3+n$$

$O(\sqrt{n}+\log(n)^2) \leq O(n+100) \leq O(n^2+\log n) \leq O(n^3+n)$

Order the following functions:

$$2^n+n^2, \quad n^4+3n, \quad n\log n+5, \quad 3^n+n^3$$

$O(n\log n+5) \leq O(n^4+3n) \leq O(2^n+n^2) \leq O(3^n+n^3)$

Order the following functions:

$$n^2+2^n, \quad n\log n+n^2, \quad \underline{n+\sqrt{n}}, \quad 10^n+n^3$$

$O(n+\sqrt{n}) \leq O(n\log n+n^2) \leq O(n^2+2^n) \leq O(10^n+n^3)$

## Alte Prüfungsaufgaben

| Claim | true | false |
|---|---|---|
| $n^3-n^2\sqrt{n} \leq O(\sqrt{n})$ | ☐ | ☒ |

$$\lim_{n\to\infty} \frac{n^{\overset{2.5}{\cancel{3}}}-n^2\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}} = \lim_{n\to\infty} n^{2.5}-n^2 = \lim_{n\to\infty} \underbrace{n^2}_{\to\infty}\cdot\underbrace{(\sqrt{n}-1)}_{\to\infty} = \infty$$

| Claim | true | false |
|---|---|---|
| $2^{(2n)} \leq O(10^n)$ | ☒ | ☐ |

$2^{(2n)} = (2^2)^n = 4^n$

$$\lim_{n\to\infty} \frac{4^n}{10^n} = \lim_{n\to\infty} \left(\frac{4}{10}\right)^n = \lim_{n\to\infty} 0.4^n = 0$$

weil $0.4 < 1$

| | true | false |
|---|---|---|
| $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \leq O(2^n)$ | $\square$ | $\boxtimes$ |

$$\lim_{n\to\infty} \frac{1\,2 \cdot 3 \cdot\,.\,.\,n}{2^n} = \lim_{n\to\infty} \frac{1}{2}\ \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2}\ \frac{5}{2} \cdot \frac{6}{2} \ldots \frac{n}{2}$$

$$= \lim_{n\to\infty} 0{,}5\ \ 1\ \ 1{,}5 \cdot 2\ \ 2{,}5\ \ 3 \ldots \frac{n}{2} = \infty$$

| | | |
|---|---|---|
| $n^{100} \leq O(2^n)$ | $\boxtimes$ | $\square$ |

Regel: Polynomiell $\leq$ exponentiell

| | | |
|---|---|---|
| $\frac{1}{2}n^2 - \sum_{i=1}^{n-1} i \leq O(n)$ | $\boxtimes$ | $\square$ |

$$= \frac{1}{2}n^2 - \frac{(n-1)((n-1)+1)}{2} = \frac{n^2 - (n-1)n}{2} = \frac{n^2 - n^2 + n}{2} = \frac{1}{2}n \leq O(n)$$

| | | |
|---|---|---|
| $n^{10} \leq O(\log(n)^{100})$ | $\square$ | $\boxtimes$ |

Regel: logarithmisch $\leq$ polynomiell

# Induktionsaufgabe

Prove by mathematical induction that the following statement holds for all $n \geq 4$:

$n! > 2^n$

**Base Case:** Sei $k = 4$

$4! = 4\ 3\ 2\ 1 = 24 > 16 = 2^4$ stimmt, deshalb gilt die Aussage für $k = 4$

**Induktions-Hypothese:** Wir nehmen an, dass die Aussage für ein beliebiges $k \in \mathbb{N}$, $k \geq 4$ gilt.

d.h. $k! > 2^k$

**Induktions-Schritt:** Wir wollen die Aussage für $k+1$ zeigen, d.h. $(k+1)! > 2^{k+1}$

$$
\begin{aligned}
(k+1)! &= (k+1)\ k! \\
&\overset{IH}{>} (k+1)\ 2^k \\
&\geq (4+1)\ 2^k \\
&> 2\ 2^k \\
&= 2^{k+1}
\end{aligned}
$$

weil $k \geq 4$

$4 + 1 = 5 > 2$

Somit ist die Aussage für alle $n \geq 4$ bewiesen per Induktion.